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LETTER TO THE EDITOR

Exact analysis of adiabatic invariants in the time-dependent harmonic oscillator

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Abstract

The theory of adiabatic invariants has a long history and important applications in physics. Here we treat exactly the general time-dependent 1D harmonic oscillator, $\ddot{q} + \omega^2(t)q = 0$, which cannot be solved in general. We follow the time-evolution of an initial ensemble of phase points with sharply defined energy E_0 and calculate rigorously the distribution of energy E_1 after time T, and all its moments, especially its average value $\bar{E}_1$ and variance μ^2. Using WKB theory, we get the exact result for the leading asymptotic behaviour of μ^2.

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Adiabatic invariants, usually denoted by I, in time-dependent dynamical systems (not necessarily Hamiltonian), are approximately conserved during a slow process of changing system parameters over a long typical time scale T. This statement is asymptotic in the sense that the conservation is exact in the limit $T \to \infty$, whilst for finite T we see the deviation $\Delta I = I_f - I_i$ of final value of I_f from its initial value I_i and would like to calculate ΔI. For the 1D harmonic oscillator it is known since Lorentz (1911) and Einstein [1] that $I = E/\omega$, which is the ratio of the total energy $E = E(t)$ and the frequency of the oscillator $\omega(t)$, both being a function of time. Of course, $2\pi I$ is exactly the area in the phase plane (q, p) enclosed by the energy contour of constant E. A general introductory account can be found in [2] and references therein, especially [3, 4].

Here, in our new approach, we re-define the old problem of adiabatic invariants in the harmonic oscillator by looking at the uniform canonical ensembles (i.e., uniform in the initial angle variable Θ) of initial conditions, all at the sharp initial energy E_0, and by studying the distribution $P(E_1)$ of the final energies E_1. The average final energy $\bar{E}_1$ also determines the variance μ^2 and all higher moments of $P(E_1)$. μ^2 goes to zero when $T \to \infty$ and we describe this asymptotic behaviour. In doing so [5], we employ the concept of the transition matrix and also our explicit WKB theory [6], which in the end provides explicit and general results in a closed form, to all orders, so far unknown in the literature.

We give a brief historical review of contributions to this field. After Einstein [1], Kulsrud [7] was the first to show, using a WKB-type method, that for a finite T, I is preserved to all orders, for the harmonic oscillator, if all derivatives of ω vanish at the beginning and at the end of the time interval, whilst if there is a discontinuity in one of the derivatives he estimated ΔI but did not give our explicit general expressions (37) and (38). Hertweck and Schlüter [8] did the same thing independently for a charged particle in a slowly varying magnetic field for infinite time domain. Kruskal, as reported in [10], and Lenard [9] studied more general systems, whilst Gardner [10] used the classical Hamiltonian perturbation theory. Courant and Snyder [11] have studied the stability of the synchrotron and analysed I employing the transition matrix. The interest then shifted to the infinite time domain. Littlewood [12] showed for the harmonic oscillator that if $\omega(t)$ is an analytic function, I is preserved to all orders of the *adiabatic parameter* $\epsilon = 1/T$. Kruskal [13] developed the asymptotic theory of Hamiltonian and other systems with all solutions nearly periodic. Lewis [14], using Kruskal's method, discovered a connection between I of the 1D harmonic oscillator and another nonlinear differential equation. Later on Symon [15] used Lewis' results to calculate the (canonical) ensemble average of I and its variance, which is the analogue of our $\bar{E}_1$ and μ^2. Finally, Knorr and Pfirsch [16] proved $\Delta I \propto \exp(-\text{const}\ \epsilon)$. Meyer [17, 18] relaxed some conditions and calculated the constant, const. Exponential preservation of I for an analytic ω on $(-\infty, +\infty)$ with constant limits at $t \to \pm\infty$, is thus well established [3].

In this work, we confine ourselves to the 1D general time-dependent harmonic oscillator, described by the Newton equation

$$\ddot{q} + \omega^2(t) q = 0 \tag{1}$$

and work out rigorously $P(E_1)$ and all its moments. Given the general $\omega(t)$ the calculation of $q(t)$ is already a very difficult and unsolvable problem. In the sense of mathematical physics (1) is exactly equivalent to the 1D stationary Schrödinger equation: the coordinate q appears instead of the probability amplitude ψ, time t appears instead of the coordinate x and $\omega^2(t)$ plays the role of $E - V(x) = \text{energy} - \text{potential}$. In this paper, we solve the above-stated problem for the general 1D harmonic oscillator, but the details will be given elsewhere [5].

We begin by defining the system by giving its Hamilton function $H = H(q, p, t)$, whose numerical value $E(t)$ at time t is precisely the total energy of the system at time t, and for the one-dimensional harmonic oscillator this is

$$H = \frac{p^2}{2M} + \frac{1}{2} M \omega^2(t) q^2, \tag{2}$$

where q, p, M, ω are the coordinate, the momentum, the mass and the frequency of the linear oscillator, respectively. The dynamics is linear in q, p, as described by (1), but nonlinear as a function of $\omega(t)$, and therefore, is subject to the nonlinear dynamical analysis. By using the index 0 and 1, we denote the initial ($t = t_0$) and final ($t = t_1$) value of the variables, and by $T = t_1 - t_0$ we denote the time interval of changing the parameters of the system.

We consider the phase-flow map (we shall call it transition map)

$$\Phi : \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \mapsto \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}. \tag{3}$$

Because equations of motion are linear in q and p, and since the system is Hamiltonian, Φ is a linear area preserving map, that is

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4}$$

with $\det(\Phi) = ad - bc = 1$. Let $E_0 = H(q_0, p_0, t = t_0)$ be the initial energy and $E_1 = H(q_1, p_1, t = t_1)$ be the final energy, that is

$$E_1 = \frac{1}{2}\left(\frac{(cq_0 + dp_0)^2}{M} + M\omega_1^2(aq_0 + bp_0)^2\right). \tag{5}$$

Introducing the new coordinates, namely the action $I = E/\omega$ and the angle ϕ,

$$q_0 = \sqrt{\frac{2E_0}{M\omega_0^2}}\cos\phi, \qquad p_0 = \sqrt{2ME_0}\sin\phi \tag{6}$$

from (5), we obtain

$$E_1 = E_0(\alpha\cos^2\phi + \beta\sin^2\phi + \gamma\sin 2\phi), \tag{7}$$

where

$$\alpha = \frac{c^2}{M^2\omega_0^2} + a^2\frac{\omega_1^2}{\omega_0^2}, \qquad \beta = \mathrm{d}^2 + \omega_1^2 M^2 b^2, \qquad \gamma = \frac{c\,\mathrm{d}}{M\omega_0} + abM\frac{\omega_1^2}{\omega_0}. \tag{8}$$

Given the uniform probability distribution of initial angles ϕ equal to $1/(2\pi)$, which defines our initial ensemble at time $t = 0$, we can now calculate the averages. Thus

$$\bar{E}_1 = \frac{1}{2\pi}\oint E_1\,\mathrm{d}\phi = \frac{E_0}{2}(\alpha + \beta). \tag{9}$$

That yields $E_1 - \bar{E}_1 = E_0(\delta\cos 2\phi + \gamma\sin 2\phi)$ and

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2}(\delta^2 + \gamma^2), \tag{10}$$

where we have denoted $\delta = (\alpha - \beta)/2$.

It follows from (8), (9) that we can also write (10) in the form

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2}\left[\left(\frac{\bar{E}_1}{E_0}\right)^2 - \frac{\omega_1^2}{\omega_0^2}\right]. \tag{11}$$

It is straightforward to show that for arbitrary positive integer m, we have $\overline{(E_1 - \bar{E}_1)^{2m-1}} = 0$ and

$$\overline{(E_1 - \bar{E}_1)^{2m}} = \frac{(2m-1)!!}{m!}(\overline{(E_1 - \bar{E}_1)^2})^m. \tag{12}$$

Thus the $2m$th moment of $P(E_1)$ is equal to $(2m-1)!!\mu^{2m}/m!$, and therefore, indeed, all moments of $P(E_1)$ are uniquely determined by the first moment $\bar{E}_1$. Obviously, $P(E_1)$ is in this sense universal, because it depends only on the average final energy $\bar{E}_1$ and the ratio ω_1/ω_0 of the final and initial frequencies, and does not depend otherwise on any details of $\omega(t)$. It has a finite support $(E_{\min}, E_{\max})$, it is an even distribution w.r.t. $\bar{E}_1 = (E_{\min} + E_{\max})/2$, and has an integrable singularity of the type $1/\sqrt{x}$ at both $E_{\min}$ and $E_{\max}$. This singularity stems from a projection of the final ensemble at t_1 onto the curves of constant final energies E_1 of $H(q, p, t_1)$. Of course, all that we say here for the distribution of energies E_1 also holds true for the final action, the adiabatic invariant $I_1 = E_1/\omega_1$. It is perhaps worthwhile to mention that the moments of our distribution according to (12) grow as $2^m/\sqrt{\pi m}$, whilst e.g. in the Gaussian distribution they grow much faster, namely, as $2^m\Gamma(m + 1/2)/\sqrt{\pi}$, where $\Gamma(x)$ denotes the gamma function.

Expression (11) is positive definite by definition and this leads to the first interesting conclusion: in full generality (no restrictions on the function $\omega(t)$) we always have $\bar{E}_1 \geqslant E_0\omega_1/\omega_0$, and therefore, the final *average* value of the adiabatic invariant $\bar{I}_1 = \bar{E}_1/\omega_1$

is always greater than or equal to the initial value $I_0 = E_0/\omega_0$. In other words, the average value of the adiabatic invariant never decreases, which is a kind of irreversibility statement. Moreover, it is constant only for infinitely slow processes $T = \infty$, which is an ideal adiabatic process, i.e. $\mu = 0$. For periodic processes $\omega_1 = \omega_0$ we see that always $\bar{E}_1 \geqslant E_0$, so the mean energy never decreases. The other extreme to $T = \infty$ is the instantaneous ($T = 0$) jump where ω_0 switches to ω_1 discontinuously, whilst q and p remain continuous, and this results in $a = d = 1$ and $b = c = 0$, and then we find

$$\bar{E}_1 = \frac{E_0}{2}\left(\frac{\omega_1^2}{\omega_0^2} + 1\right), \qquad \mu^2 = \frac{E_0^2}{8}\left[\frac{\omega_1^2}{\omega_0^2} - 1\right]^2. \tag{13}$$

Below we shall treat the special case with $\omega_1^2 = 2\omega_0^2$, and thus will find $\mu^2 / E_0^2 = 1/8$.

Our general study now focuses on the calculation of the transition map (4), namely, its matrix elements a, b, c, d. Starting from the Hamilton function (2) and its Newton equation (1), we consider two linearly independent solutions $\psi_1(t)$ and $\psi_2(t)$ and introduce the matrix

$$\Psi(t) = \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ M\dot{\psi}_1(t) & M\dot{\psi}_2(t) \end{pmatrix}. \tag{14}$$

Consider a solution $\hat{q}(t)$ of (1), such that

$$\hat{q}(t_0) = q_0, \qquad \dot{\hat{q}}(t_0) = p_0/M. \tag{15}$$

Because ψ_1 and ψ_2 are linearly independent, we can look for $\hat{q}(t)$ in the form

$$\hat{q}(t) = A\psi_1(t) + B\psi_2(t). \tag{16}$$

Then A and B are determined by

$$\begin{pmatrix} A \\ B \end{pmatrix} = \Psi^{-1}(t_0)\begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \tag{17}$$

Let $q_1 = \hat{q}(t_1)$, $p_1 = M\dot{\hat{q}}(t_1)$. Then from (15)–(17), we see that

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \Psi(t_1)\Psi^{-1}(t_0)\begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \tag{18}$$

We recognize the matrix on the right-hand side of (18) as the transition map Φ, that is

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Psi(t_1)\Psi^{-1}(t_0). \tag{19}$$

Due to lack of space, we mention only linear model: $\omega^2(t)$: ω_0^2 for $t \leqslant 0$, $\omega_0^2 + (t/T)\left(\omega_1^2 - \omega_0^2\right)$ for $0 < t < T$ and ω_1^2 for $t \geqslant T$. Equation (1) can be solved exactly in terms of the Airy functions yielding $\bar{E}_1, \mu^2$ etc. As an example, if $\omega_0^2 = 1$ and $\omega_1^2 = 2$, μ^2 goes correctly from $1/8$ at $T = 0$ to zero as $T \to \infty$, in a typical oscillatory way. Using the asymptotic expressions for the Airy functions, we find the leading asymptotic approximation

$$\frac{\mu^2}{E_0^2} = \frac{\overline{(E_1 - \bar{E}_1)^2}}{E_0^2} \approx \frac{\epsilon^2}{128}\left(9 - 4\sqrt{2}\cos\left(\frac{4 - 8\sqrt{2}}{3\epsilon}\right)\right) \qquad (\epsilon = 1/T). \tag{20}$$

We proceed with the calculation of the transition map Φ in the general case, and because (1) is generally not solvable, we have ultimately to resort to some approximations. Since the adiabatic limit $\epsilon \to 0$ is the asymptotic regime that we would like to understand, the application of the rigorous WKB theory (up to all orders) is most convenient, and usually it turns out that the leading asymptotic terms are well described by just the leading WKB terms.

We introduce re-scaled and dimensionless time $\lambda = \epsilon t$, so that (1) is turned to

$$\epsilon^2 q''(\lambda) + \omega^2(\lambda) q(\lambda) = 0. \tag{21}$$

Let $q_+(\lambda)$ and $q_-(\lambda)$ be two linearly independent solutions of (21). Then the matrix (14) takes the form

$$\Psi_\lambda = \begin{pmatrix} q_+(\lambda) & q_-(\lambda) \\ \epsilon M q_+'(\lambda) & \epsilon M q_-'(\lambda) \end{pmatrix} \tag{22}$$

and taking into account that $\lambda_0 = \epsilon t_0$, $\lambda_1 = \epsilon t_1$, we obtain for the matrix (19) the expression $\Phi = \Psi_\lambda(\lambda_1)\Psi_\lambda^{-1}(\lambda_0)$. We now use the WKB method in order to obtain the coefficients a, b, c, d of the matrix Φ.[1] To do so, we look for solution of (21) in the form $q(\lambda) = w \exp\{\sigma(\lambda)/\epsilon\}$, where $\sigma(\lambda)$ is a complex function that satisfies the differential equation

$$(\sigma'(\lambda))^2 + \epsilon\sigma''(\lambda) = -\omega^2(\lambda) \tag{23}$$

and w is some constant with dimension of length. The WKB expansion for the phase is $\sigma(\lambda) = \sum_{k=0}^{\infty} \epsilon^k \sigma_k(\lambda)$. Substituting this expression into (23) and comparing like powers of ϵ gives the recursion relation

$$\sigma_0'^2 = -\omega^2(\lambda), \qquad \sigma_n' = -\frac{1}{2\sigma_0'}\left(\sum_{k=1}^{n-1} \sigma_k'\sigma_{n-k}' + \sigma_{n-1}''\right). \tag{24}$$

Here, we apply our WKB notation and formalism [6] and we can choose $\sigma_{0,+}'(\lambda) = \mathrm{i}\omega(\lambda)$ or $\sigma_{0,-}'(\lambda) = -\mathrm{i}\omega(\lambda)$. That results in two linearly independent solutions of (21) given by the WKB expansions with the coefficients

$$\sigma_{0,\pm}(\lambda) = \pm\mathrm{i}\int_{\lambda_0}^{\lambda} \omega(x)\,\mathrm{d}x, \qquad \sigma_{1,\pm}(\lambda) = -\frac{1}{2}\log\frac{\omega(\lambda)}{\omega(\lambda_0)},$$

$$\sigma_{2,\pm} = \pm\frac{\mathrm{i}}{8}\int_{\lambda_0}^{\lambda} \frac{3\omega'(x)^2 - 2\omega(x)\omega''(x)}{\omega(x)^3}\,\mathrm{d}x, \ldots$$

Since $\omega(\lambda)$ is a real function we deduce from (24) that all functions σ_{2k+1}' are real and all functions σ_{2k}' are purely imaginary and $\sigma_{2k,+}' = -\sigma_{2k,-}'$, $\sigma_{2k+1,+}' = \sigma_{2k+1,-}'$, where $k = 0, 1, 2, \ldots$, and thus we have $\sigma_+' = A(\lambda) + \mathrm{i}B(\lambda)$, $\sigma_-' = A(\lambda) - \mathrm{i}B(\lambda)$, where $A(\lambda) = \sum_{k=0}^{\infty} \epsilon^{2k+1}\sigma_{2k+1}'(\lambda)$, $B(\lambda) = -\mathrm{i}\sum_{k=0}^{\infty} \epsilon^{2k}\sigma_{2k,+}'(\lambda)$. Integration of the above equations yields $\sigma_+ = r(\lambda) + \mathrm{i}s(\lambda)$, $\sigma_- = r(\lambda) - \mathrm{i}s(\lambda)$, where $r(\lambda) = \int_{\lambda_0}^{\lambda} A(x)\,\mathrm{d}x$, $s(\lambda) = \int_{\lambda_0}^{\lambda} B(x)\,\mathrm{d}x$. Below we shall denote $s_1 = s(\lambda_1)$.

Using this notation, we find that the elements of the transition matrix Φ have the following form, after taking into account that $\det(\Phi) = ab - cd = 1$,

$$a = -\frac{1}{\sqrt{B_0 B_1}}\left[A_0 \sin\left(\frac{s_1}{\epsilon}\right) - B_0 \cos\left(\frac{s_1}{\epsilon}\right)\right], \qquad b = \frac{1}{M\sqrt{B_0 B_1}} \sin\left(\frac{s_1}{\epsilon}\right),$$
$$c = -\frac{M}{\sqrt{B_0 B_1}}\left[(A_0 A_1 + B_0 B_1)\sin\left(\frac{s_1}{\epsilon}\right) + (A_0 B_1 - A_1 B_0)\cos\left(\frac{s_1}{\epsilon}\right)\right], \tag{25}$$
$$d = \frac{1}{\sqrt{B_0 B_1}}\left[A_1 \sin\left(\frac{s_1}{\epsilon}\right) + B_1 \cos\left(\frac{s_1}{\epsilon}\right)\right].$$

This is so far exact result, based on the WKB-expansion technique. What we are mostly interested in is the asymptotic behaviour of μ^2 when ϵ is small and tends to zero.

[1] There is a substantial literature on the WKB method, which due to limited space cannot be reviewed here. But we should mention the classic works by Fröman and Fröman, who have found a number of interesting relationships, e.g. a relation between the even and odd order terms [19], although we do not use it here, so that our exposition is self-contained.

Let us consider the first-order WKB approximation, that is

$$A(\lambda) \approx \epsilon\sigma'_{1,+}(\lambda), \qquad B(\lambda) \approx \frac{\sigma'_{0,+}(\lambda)}{\mathrm{i}} = \omega(\lambda). \tag{26}$$

We find for the variance (11)

$$\frac{\mu^2}{E_0^2} = \epsilon^2\left(\frac{\omega_1^2\omega_0'^2}{8\omega_0^6} + \frac{\omega_1'^2}{8\omega_0^2\omega_1^2} - \frac{\omega_0'\omega_1'}{4\omega_0^4}\cos\left(\frac{2}{\epsilon}\int_{\lambda_0}^{\lambda_1}\omega(x)\,\mathrm{d}x\right)\right) + O(\epsilon^3). \tag{27}$$

Substituting into (27) $\omega(\lambda) = \sqrt{1+\lambda}$, we obtain exactly the approximation (20).

Suppose now that all derivatives at λ_0 and λ_1 vanish up to order $(n-1)$, i.e. $\omega'(\lambda_0) = \omega'(\lambda_1) = \cdots = \omega^{(n-1)}(\lambda_0) = \omega^{(n-1)}(\lambda_1) = 0$ and $\omega^{(n)}(\lambda_0)\omega^{(n)}(\lambda_1) \neq 0$. Then $\sigma_1'(\lambda_0) = \sigma_1'(\lambda_1) = \cdots = \sigma'_{n-1}(\lambda_0) = \sigma'_{n-1}(\lambda_1) = 0$, $\sigma'_n(\lambda_0)\sigma_n(\lambda_1) \neq 0$.

Hence, in the case $n = 2k-1$ we can assume

$$A(\lambda) = \epsilon^{2k-1}\sigma'_{2k-1,+}(\lambda) + \text{h.o.t.}, \qquad B(\lambda) = \omega(\lambda) - \mathrm{i}\epsilon^{2k}\sigma'_{2k,+}(\lambda) + \text{h.o.t.} \tag{28}$$

and obtain

$$\frac{\mu^2}{E_0^2} = \epsilon^{4k-2}\left(\frac{\sigma'_{2k-1,+}(\lambda_1)^2}{2\omega_0^2} + \frac{\omega_1^2\sigma'_{2k-1,+}(\lambda_0)^2}{2\omega_0^4} - \frac{\omega_1\sigma'_{2k-1,+}(\lambda_0)\sigma'_{2k-1,+}(\lambda_1)}{\omega_0^3}\cos\left(\frac{2s_1}{\epsilon}\right)\right) + O(\epsilon^{4k-1}). \tag{29}$$

In the case when $n = 2k$, we can suppose

$$A(\lambda) = \epsilon^{2k+1}\sigma'_{2k+1,+}(\lambda) + \text{h.o.t.}, \qquad B(\lambda) = \omega(\lambda) - \mathrm{i}\epsilon^{2k}\sigma'_{2k,+}(\lambda) + \text{h.o.t.} \tag{30}$$

Then, similarly as above, we obtain

$$\frac{\mu^2}{E_0^2} = -\epsilon^{4k}\left(\frac{\sigma'_{2k,+}(\lambda_1)^2}{2\omega_0^2} + \frac{\omega_1^2\sigma'_{2k,+}(\lambda_0)^2}{2\omega_0^4} - \frac{\omega_1\sigma'_{2k,+}(\lambda_0)\sigma'_{2k,+}(\lambda_1)}{\omega_0^3}\cos\left(\frac{2s_1}{\epsilon}\right)\right) + O(\epsilon^{4k+1}). \tag{31}$$

From this we can conclude that if $\omega(t)$ is of class $\mathcal{C}^m$ (having m continuous derivatives, $m = n-1$) μ^2 goes to zero oscillating but in the mean as $\propto \epsilon^{2n} = \epsilon^{2(m+1)}$. If $\omega(t)$ is an analytic function on the real-time axis $(-\infty, +\infty)$, the decay to zero is oscillating and on the average is exponential $\propto \exp(-\text{const }\epsilon)$ [3, 16–18]. This exponential smallness stems from the divergence of the relevant series and has been extensively studied in related works [20–24], where the resummation techniques have been devised.

Our method also enables us to easily calculate higher terms in (27) and in the general equations (29) and (31) [5].

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